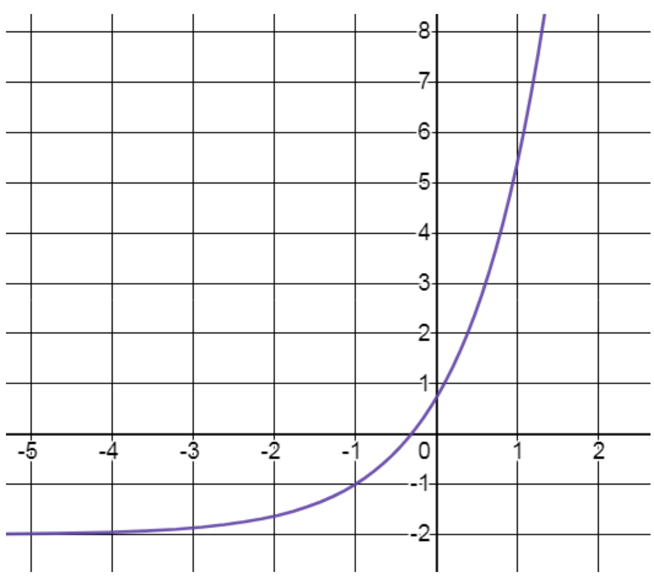
GRAPHING WITH BASE *e*

Since *e* is a positive real number not equal to 1 (it's greater than 1), it's allowed to be the base of an exponential function. In fact, the base-*e* exponential function, $y = e^x$, is one of the most important exponential functions in business, biology, chemistry, ecology, statistics, physics, and electronics. Let's graph one of these.

EXAMPLE 1: **Graph:** $y = e^{x+1} - 2$

Solution: Use your calculator to verify that each of the following ordered pairs is on the graph of the function:

(–4, –1.95) (–3, –1.86) (–2, –1.63) (–1, –1) (0, 0.72) (1, 5.39)



Exponential Growth

The **domain** of the function is $\mathbb{R}$. To determine the range, we need to verify what the graph appears to show, that the line $y = -2$ is a horizontal asymptote. Let x be a large negative number, for instance $x = -20$. Then the y-value is

$$y = e^{-20+1} - 2 = e^{-19} - 2 = .000000006 - 2 = -\mathbf{1.999999994}$$

which is <u>extremely</u> close to (and slightly above) -2. We can be pretty confident that the **horizontal asymptote** is $\boldsymbol{y = -2}$. From this fact, we can conclude that the **range** of the function is all real numbers greater than -2: $\mathbf{(-2, \infty)}$.

For our last bit of analysis, we summarize the behavior of the function at extreme values of x by writing the limits

$$\lim_{x \to \infty} (e^{x+1} - 2) = \infty \qquad \text{and} \qquad \lim_{x \to -\infty} (e^{x+1} - 2) = -\mathbf{2}$$

The first of these two limits indicates that the graph is **increasing**; that is, as you move from left to right along the graph, the graph grows taller and taller (and never levels off). We can call this **exponential growth**. In our following example, we change one little (but critical) part of the equation (which part?) and we get a curve which represents **exponential decay**.

<u>EXAMPLE 2:</u> **Graph:** $\boldsymbol{y = e^{-x} + 1}$

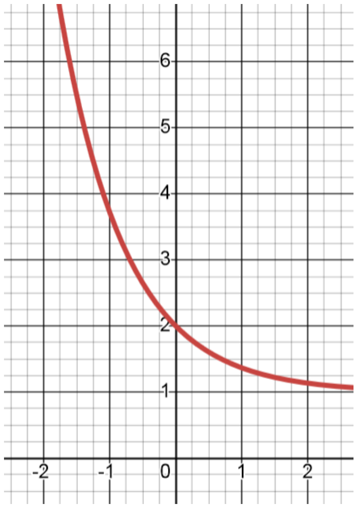
<u>Solution:</u> Let's go for the y-intercept first. Letting $x = 0$ produces $y = e^{-0} + 1 = e^{0} + 1 = 1 + 1 = 2$. The y-intercept is therefore **(0, 2)**. Use your calculator to verify some other points on the graph:

(–3, 21.09) (–2, 8.39) (–1, 3.72) (1, 1.37) (2, 1.14) (3, 1.05)

To emphasize the limits as x approaches ∞ or $-\infty$, let's calculate two more points:

(–10, 22027.47) (10, 1.000045)

It appears that as $x \to -\infty$, $y \to \infty$. Also, as $x \to \infty$, $y \to 1$.



Exponential Decay

The domain of the function is ℝ, simply because any real number x can be used in the formula without a problem.

The range is all real numbers greater than 1: **(1, ∞)**

From the graph and the points we calculated, we see that there's a horizontal asymptote at $\boldsymbol{y = 1}$.

> In computer languages, the function $y = e^x$ cannot be written that way since computer code cannot contain an exponent. So instead we write
>
> $$y = \exp(x).$$

Homework

Graph each of these exponential functions:

1. $y = e^x$	2. $f(x) = e^{-x}$	3. $g(x) = e^2 + 2$
4. $h(x) = e^{x-2}$	5. $f(x) = e^x - 1$	6. $y = e^{x-1} + 2$
7. $E(x) = e^{x+2} - 3$	8. $y = e^{-2x}$	9. $y = e^{-x} - 1$
10. $h(x) = e^{-x} + 2$	11. Describe the graph of $y = ex + e$.	

❑ *THE BELL-SHAPED CURVE*

Some of you will take Statistics soon, and you will learn that the most important function in the course is a special bell-shaped curve called the ***standard normal*** curve (or Gaussian Curve). It's an exponential function, and its formula is a bit complicated, but we have all the tools needed to find some of its points and then sketch it.

The formula for the standard normal curve is given by

$$\boxed{y = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}}$$

Note that this function is a type of ***exponential function*** because the base is a constant (in this case, e) and the variable x occurs in the exponent. Let's decipher this formula before we try to plot any points: The coefficient of the exponential function is $\frac{1}{\sqrt{2\pi}}$, the base is e, and the exponent on the e is $-\frac{1}{2}x^2$.

Let's calculate some (x, y) pairs for this function and see what kind of graph we get. Let's start by letting $x = 0$. This, of course, will yield the y-intercept:

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}}e^0 = \frac{1}{\sqrt{2\pi}}(1) \approx 0.3989$$

This gives us the approximate point **(0, 0.4)**. We're off to a good start.

Now we'll choose $x = 1$. Plugging this number into the function gives

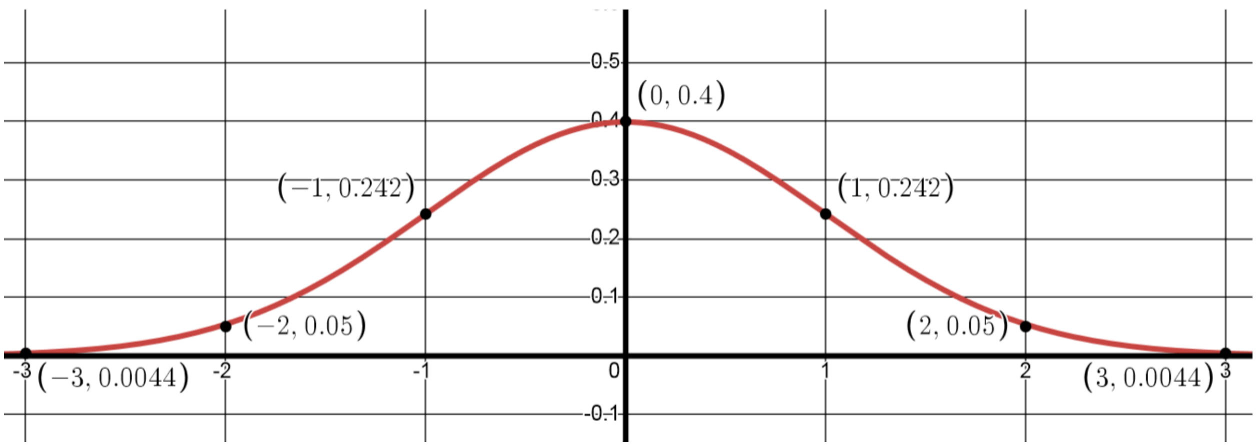
$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}(0.6065) \approx 0.2420$$

Our second point is therefore **(1, 0.24)**. Now for $x = -1$. You should be able to do this one in your head. First, the –1 is squared, giving 1 — but this is exactly what we had in the previous calculation, and so the y-value must also be 0.2420. Our third point is therefore **(–1, 0.24)**.

Choosing $x = 2$ (or -2) gives the following calculation for y:

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\pm 2)^2} = \frac{1}{\sqrt{2\pi}} e^{-2} = \frac{1}{\sqrt{2\pi}}(0.1353) \approx 0.0540$$

We now have two more points for our graph: **(2, 0.05)** and **(–2, 0.05)**. Continuing in this manner, we can find the points **(3, 0.0044)** and **(–3, 0.0044)**. It appears that as we get farther from the origin, the y-values are getting smaller and smaller, but always remaining



positive. If we plot the seven points we've just computed and then connect them with a smooth curve, we get the following:

What can we infer from this graph? Lots of stuff:

A. The **domain** is $\mathbb{R}$, all real numbers. [The graph goes infinitely to the right and infinitely to the left.] This means that x can take on any value at all and the formula will work out just fine.

B. The **range** is all numbers that are bigger than 0 but less than or equal to *approximately* 0.4. In interval notation: **(0, 0.4]**.

C. If we fold the portion of the graph in Quadrant I over into Quadrant II, the curves will coincide. We therefore have ***y*-axis symmetry**. [You may have learned to change x to $-x$ and see if you get the same formula; you do.]

D. The calculations and the graph indicate these two limits:

$$\lim_{x \to \pm\infty} y = \mathbf{0}$$

E. And from the limits we deduce that the line $\boldsymbol{y} = \mathbf{0}$ (the x-axis) is a **horizontal asymptote** of the graph.

F. One reason for some of the numbers in the function is that we want the total **area** between the curve and the x-axis to be exactly **1**.

Review Problems

12. a. List the domain, range, intercepts, and asymptotes of $y = e^x$.

b. $\lim\limits_{x \to \infty} e^x$ c. $\lim\limits_{x \to -\infty} e^x$ d. $\lim\limits_{x \to 0} e^x$

13. Let $f(x) = e^x$ and $g(x) = e^{x+\pi} + 100$. Describe the graph of g relative to the graph of f.

14. Graph: $y = e^{x-2} - 3$. Calculate the y-intercept and any asymptotes.

15. Graph: $y = e^{-x} - 3$. Calculate the y-intercept and any asymptotes.

16. Graph: $N(x) = e^{-x^2}$. Hint: If $x = 1$, then $e^{-1^2} = e^{-1} = \frac{1}{e} \approx 0.3679$.
Prove that the graph has y-axis symmetry.

17. Explain why the graph of $y = x^2 + ex + \pi$ is a parabola, and then prove that it has no x-intercepts.

18. True/False:

a. $\lim_{x\to\infty} f(x) = L$ means the same thing as "As $x \to \infty$, $f(x) \to L$."

b. If $P(x) = \dfrac{6}{1+x^2}$, then $\lim_{x\to 0} P(x) = 6$.

c. The function $y = e^x$ is a polynomial.

d. The domain of the above function is $\mathbb{R}$.

e. The range of the above function is $(0, \infty)$.

f. Compared to the graph of $y = e^x$, the graph of $y = e^x + 2$ is two units higher.

g. The function $y = e^{-x}$ is decreasing.

h. The function $y = e^{-x}$ has a horizontal asymptote.

i. The function $y = e^{-x}$ has a vertical asymptote.

j. The slope of the line $y = \pi x + e$ is e.

k. The y-intercept of the graph of

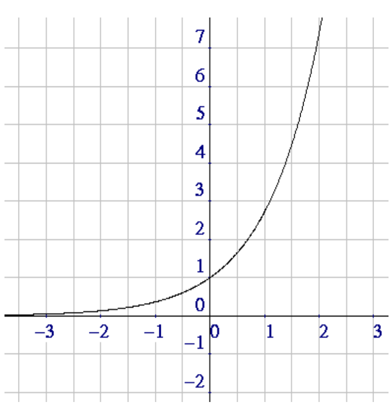
$$h(x) = e^{1-x} + 2$$

is $(0, e + 2)$.

l. $e > \pi$

Solutions

1.



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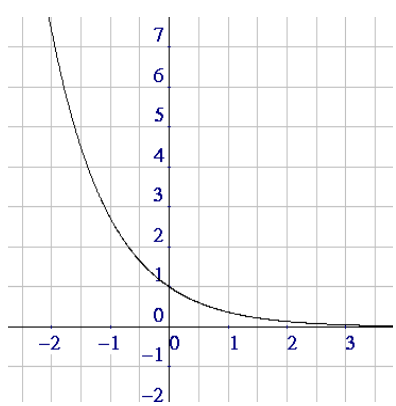
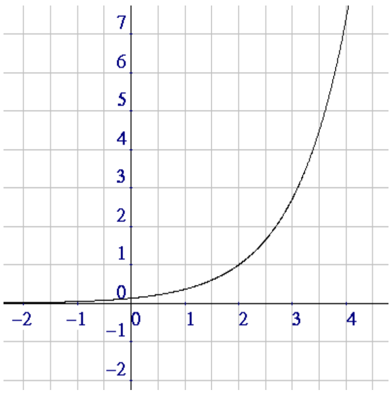


3. I'd like to see what you think the graph is.

4.



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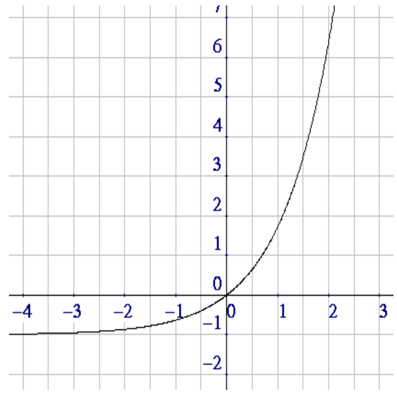

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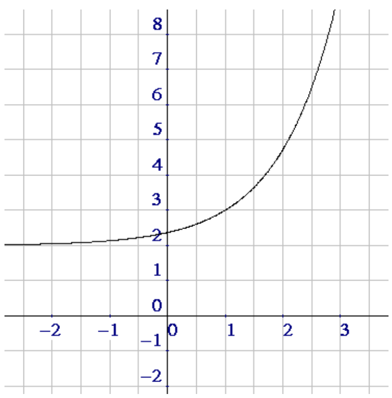

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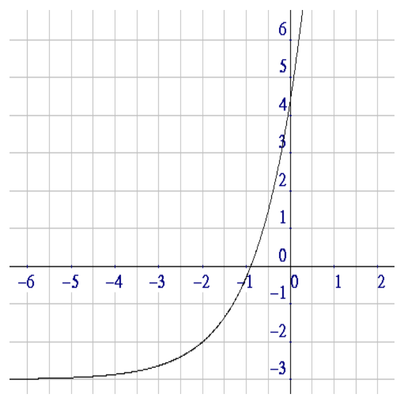

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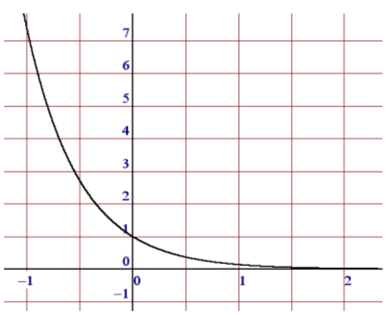

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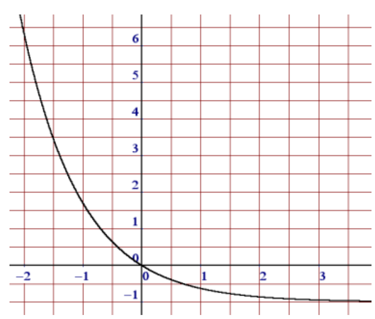

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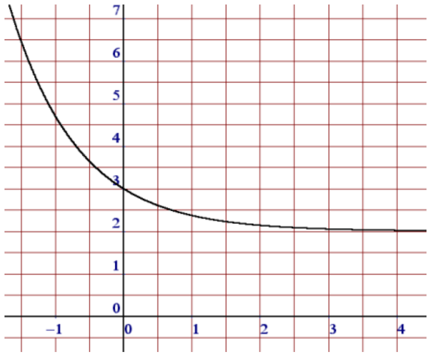


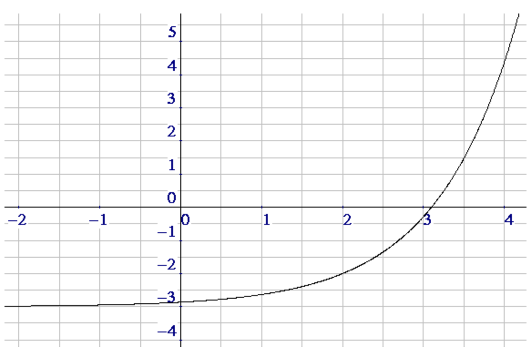
11. The graph of $y = ex + e$ is a straight line with slope e and having a y-intercept of $(0, e)$.

12. a. Domain = $\mathbb{R}$; Range = $(0, \infty)$;
x-int: none; y-int: $(0, 1)$;
vert asy: none; horiz asy: $y = 0$

b. ∞ c. 0 d. 1

13. The graph of g is the graph of f shifted π units to the left and 100 units up.

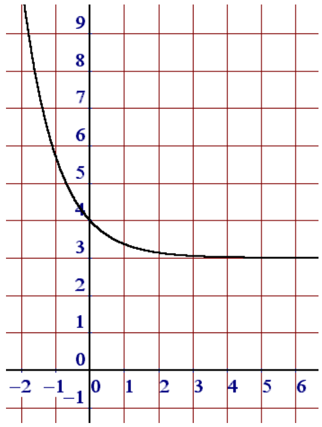
14.



y-int: $(0, e^{-2} - 3) \approx (0, -2.86)$

horiz asy: $y = -3$

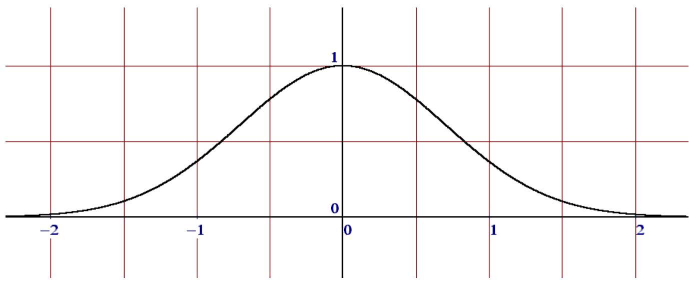
15.



y-int: (0, 4)

horiz asy: $y = 3$

16.



To prove y-axis symmetry, we change each x in the formula to $-x$ and see what we get: $e^{-(-x)^2} = e^{-x^2}$, the same formula. We're done.

17. It's a parabola because it fits the form $y = ax^2 + bx + c$; in fact, $a = 1$, $b = e$, and $c = \pi$.
To find x-intercepts, set y to 0; solve the resulting quadratic equation for x using the Quadratic Formula. You should get a negative radicand, indicating no solutions in $\mathbb{R}$, and thus no x-intercepts.

18. a. T b. T c. F d. T e. T f. T g. T h. T
i. F j. F k. T l. F

"The direction in which education starts a man will determine his future life." Plato